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Interfacial Tension of the Chiral Potts Model

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We calculate the interfacial tension of the N-state chiral Potts model by solving the functional relations for the transfer matrices of the model with skewed boundary conditions. Our result is valid for the general physical model (with positive Boltzmann weights) and at all subcritical temperatures. The interfacial tension has been calculated previously for the superintegrable chiral Potts model with skewed boundary conditions. Using Z-invariance, Baxter has argued that the interfacial tension of this model should be the same as the interfacial tension of the general physical model. We show that this is indeed the case.

KEY WORDS: Statistical mechanics; lattice models; chiral Potts model; interfacial tension.

1. INTRODUCTION

The interfacial tensions of the chiral Potts model have been calculated recently in a series of papers by Baxter.[1,2] In the first, the functional relations which were originally derived for the model with periodic boundary conditions in refs. 3 and 4 were rederived for the model with skewed boundary conditions. The interfacial tensions were then calculated for the model in the zero-temperature limit. They were found to be independent of the vertical rapidities, as was expected from Z-invariance.[5]

This independence implied a shortcut to calculating the interfacial tensions for the model at arbitrary temperatures; one could choose the vertical rapidities so as to make the model "superintegrable." The superintegrable chiral Potts model is a nonphysical model corresponding to a particular choice of the vertical rapidities. This simplifies the functional relations and

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hence the transfer matrices, making the model easier to solve. The interfacial tensions thus obtained should be the same as those of the physical model. This program was carried through in ref. 2 and the interfacial tensions and critical exponents calculated.

There is, however, a potential flaw in this argument. As the superintegrable chiral Potts model is nonphysical, there is a problem in applying Z-invariance, although in refs. 1 and 2 it was argued that the result for the interfacial tensions should still hold.

In this paper, we directly calculate the interfacial tensions, using the low-temperature results of ref. 1, but without reference to the superintegrable model. We consider the functional relations of the general physical model with skewed boundary conditions at general subcritical temperatures. We find that the method of solving the functional relations in refs. 6 and 7 for the model with periodic boundary conditions generalizes with only minor modifications to the skewed case. We solve the functional relations for a band of L complex largest eigenvalues, and from these we calculate the free energy and interfacial tension. We find that this is indeed the result given in ref. 2 and hence that the Z-invariance arguments hold.

In refs. 6 and 7 the free energy was calculated for the homogeneous model (with all vertical rapidities equal), and for $|\lambda_q| < 1$. Here we consider an alternating model (with the vertical rapidities alternately p and p' along the row), and with $|\lambda_q| > 1$. We find our result to be the analytical continuation of the $|\lambda_q| < 1$ result, and that the free energy of the alternating model is simply the arithmetic mean of the free energy of the homogeneous model with vertical rapidities p and p', as we expect from Z-invariance.[5]

2. THE MODEL

Consider the square $L \times M$ lattice rotated through 45 deg, shown in Fig. 1. It has $2M$ rows, with L sites in each row, for a total of $2LM$ sites. Spins, denoted σ_i, sit on the sites of the lattice, and take on the integer values $0, 1, \ldots, N-1$.

We impose the skewed boundary conditions used in ref. 1 in the horizontal direction, that is,

$$\sigma_{L+1} = \sigma_1 - r \tag{2.1}$$

where σ_1 and σ_{L+1} are, respectively, the first and last spins in each row, and retain normal periodic boundary conditions vertically. Here r (the "skew parameter") is some integer, restricted to the interval $0 \leqslant r \leqslant N-1$. When $r = 0$, we regain normal periodic boundary conditions.

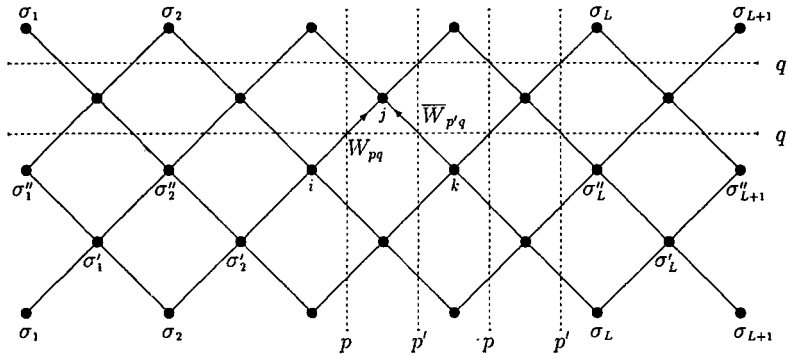

Fig. 1. The square lattice with L sites per row and $M=2$. The Boltzmann weights W_{pq} and $\bar{W}_{p'q}$ in the SW → NE and SE → NW directions, respectively, are indicated. Also shown are the horizontal rapidity variable q and the alternating vertical rapidities p and p'.

Let ω be the primitive root of unity $\omega = e^{2\pi i/N}$, and let $\omega^{1/2} = e^{\pi i/N}$. Let k and k' be real parameters, $0 \leqslant k,\ k' \leqslant 1$, related by

$$k^2 + k'^2 = 1 \tag{2.2}$$

and let η be the real solution to

$$\eta^N = (1-k')/(1+k') \tag{2.3}$$

Further, define the "q-variables" $q = \{x_q, y_q, \mu_q, \lambda_q, t_q\}$ to be a set of complex numbers related by

$$x_q^N + y_q^N = k(1 + x_q^N y_q^N), \qquad k x_q^N = 1 - k'\mu_q^{-N}, \qquad k y_q^N = 1 - k'\mu_q^N \tag{2.4}$$

$$t_q = x_q y_q, \qquad \lambda_q = \mu_q^N \tag{2.5}$$

The variables t_q and λ_q are related by

$$\lambda_q + \frac{1}{\lambda_q} = \frac{1 + k'^2 - k^2 t_q^N}{k'}, \qquad \left(\frac{\lambda_q - 1}{\lambda_q + 1}\right)^2 = \frac{\eta^N - t_q^N}{\eta^{-N} - t_q^N} \tag{2.6}$$

Sets of p- and p'-variables are defined analogously. Due to the relations between the variables, there is only one independent parameter in each of these sets. Of the various automorphisms of the variables that leave these relations unchanged (see ref. 4), we use the following:

$$R\colon q \to \{y_q, \omega x_q, \mu_q^{-1}, \omega t_q, \lambda_q^{-1}\} \tag{2.7}$$

$$\bar{q}(k, l)\colon q \to \{\omega^k y_q, \omega^l x_q, \mu_q^{-1}, \omega^{k+l} t_q, \lambda_q^{-1}\} \tag{2.8}$$

where $R = \bar{q}(0, 1)$. We shall often write $\bar{q}(k, l)$ as $\bar{q}kl$.

Associated with each of the dashed lines in Fig. 1 is a rapidity variable, we call it q for the horizontal and p for the vertical lines. In general we can allow them to be different for each line, but a "sufficient level of generality" is to set all of the horizontal rapidities to the same value q and to allow the vertical ones to alternate along the row, taking the values p and p' as indicated in the figure.

Nearest neighbor spins on the lattice interact along SW $\to$ NE edges between sites i and j with Boltzmann weight $W_{pq}(\sigma_i - \sigma_j)$, where (for n an integer)

$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^{n} \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q} \tag{2.9}$$

and on SE $\to$ NW edges, the spins between sites k and j interact with Boltzmann weight $\bar{W}_{p'q}(\sigma_k - \sigma_j)$, where

$$\bar{W}_{p'q}(n) = (\mu_{p'}\mu_q)^n \prod_{j=1}^{n} \frac{\omega x_{p'} - \omega^j x_q}{y_q - \omega^j y_{p'}} \tag{2.10}$$

[We have normalized the weights so that $W(0) = \bar{W}(0) = 1$. This is the normalization used in the functional relations in ref. 1, but is different from that used in refs. 6 and 7 for the calculation of the free energy.] The weights satisfy the periodicity conditions

$$W_{pq}(n+N) = W_{pq}(n) \qquad \text{and} \qquad \bar{W}_{p'q}(n+N) = \bar{W}_{p'q}(n) \tag{2.11}$$

which follow from the identities

$$\left(\frac{\mu_p}{\mu_q}\right)^N = \frac{y_p^N - x_q^N}{y_q^N - x_p^N} \qquad \text{and} \qquad (\mu_p\mu_q)^N = \frac{y_q^N - y_p^N}{x_p^N - x_q^N} \tag{2.12}$$

The Boltzmann weights also satisfy the star–triangle relation

$$\sum_{d=0}^{N-1} \bar{W}_{qp'}(b-d)\, W_{pp'}(a-d)\, \bar{W}_{pq}(d-c)$$
$$= (f_{pq} f_{qp'} / f_{pp'})\, W_{pq}(a-b)\, \bar{W}_{pp'}(b-c)\, W_{qp'}(a-c) \tag{2.13}$$

for all rapidities p, p', q and for all integers a, b, c. The function f_{pq} is a complex-valued function of the p- and q-variables, its Nth power being

$$f_{pq}^N = \det{}_N[\bar{W}_{pq}(i-j)] \Big/ \prod_{n=0}^{N-1} W_{pq}(n) \tag{2.14}$$

where in our normalization

$$\det{}_N[\bar{W}_{pq}(i-j)] = N^{N/2} e^{i\pi(N-1)(N-2)/12} \prod_{j=1}^{N-1} \frac{(t_p - \omega^j t_q)^j}{(x_p - \omega^j x_q)^j (y_p - \omega^j y_q)^j} \tag{2.15}$$

We also define a related function g_{pq}, with its Nth power being

$$g_{pq}^N = \det{}_N[\bar{W}_{pq}(i-j)] \prod_{n=0}^{N-1} W_{pq}(n) \tag{2.16}$$

The functions f_{pq} and g_{pq} satisfy the following relations:

$$f_{pp} = g_{pp} = 1, \qquad f_{pq} f_{qp} = g_{pq} g_{qp} = N \frac{(x_p - x_q)(y_p - y_q)(t_p^N - t_q^N)}{(x_p^N - x_q^N)(y_p^N - y_q^N)(t_p - t_q)} \tag{2.17}$$

The row-to-row transfer matrices T and $\hat{T}$ are defined as follows. Let the spins in three consecutive rows, each above the other, be $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_L\}$, $\sigma' = \{\sigma_1', \sigma_2', ..., \sigma_L'\}$, and $\sigma'' = \{\sigma_1'', \sigma_2'', ..., \sigma_L''\}$. These are the bottom three rows of the lattice in Fig. 1. Then T is the N^L by N^L matrix with entries

$$T_{\sigma\sigma'} = \prod_{j=1}^{L} W_{pq}(\sigma_j - \sigma_j') \, \bar{W}_{p'q}(\sigma_{j+1} - \sigma_j') \tag{2.18}$$

and $\hat{T}$ is the N^L by N^L matrix with entries

$$\hat{T}_{\sigma'\sigma''} = \prod_{j=1}^{L} \bar{W}_{pq}(\sigma_j' - \sigma_j'') \, W_{p'q}(\sigma_j' - \sigma_{j+1}'') \tag{2.19}$$

We will regard p and p' as fixed, q as a free variable, and denote the dependence of T and $\hat{T}$ on q explicitly by T_q and $\hat{T}_q$.

As a result of the star–triangle relation, the transfer matrices satisfy the commutation relation

$$T_q \hat{T}_s = (f_{p'q} f_{ps} / f_{pq} f_{p's})^L \, T_s \hat{T}_q \tag{2.20}$$

for all horizontal rapidities q and s, and for any skew parameter r.

The commutation relation implies that one can simultaneously diagonalize the transfer matrices T_q and $\hat{T}_q$ by the coupled similarity transformations

$$T_q \to P^{-1} T_q Q, \qquad \hat{T}_q \to Q^{-1} \hat{T}_q P \tag{2.21}$$

where P and Q are matrices which are independent of the q-variables.

The eigenvectors of T_q and $\hat{T}_q$, $\mathbf{x}$ and $\mathbf{y}$, are the solutions to the coupled vector equations

$$T_q\mathbf{y} = (\text{scalar})\,\mathbf{x}, \qquad \hat{T}_q\mathbf{x} = (\text{scalar})\,\mathbf{y} \tag{2.22}$$

where the eigenvectors are independent of q also.

Multiplying any equation involving the matrices T_q or $\hat{T}_q$ (or any of the τ_j matrices, which are defined shortly) on the right by the appropriate $\mathbf{x}$ or $\mathbf{y}$ effectively replaces the matrix by its eigenvalue. Thus any relation between the matrices can also be considered as a relation between their eigenvalues, and hence our notation does not differentiate between the matrices and their eigenvalues.

The transfer matrices T_q and $\hat{T}_q$ are not completely independent. From Fig. 1, we see that $\hat{T}_q$ can be obtained from T_q by interchanging p and p'

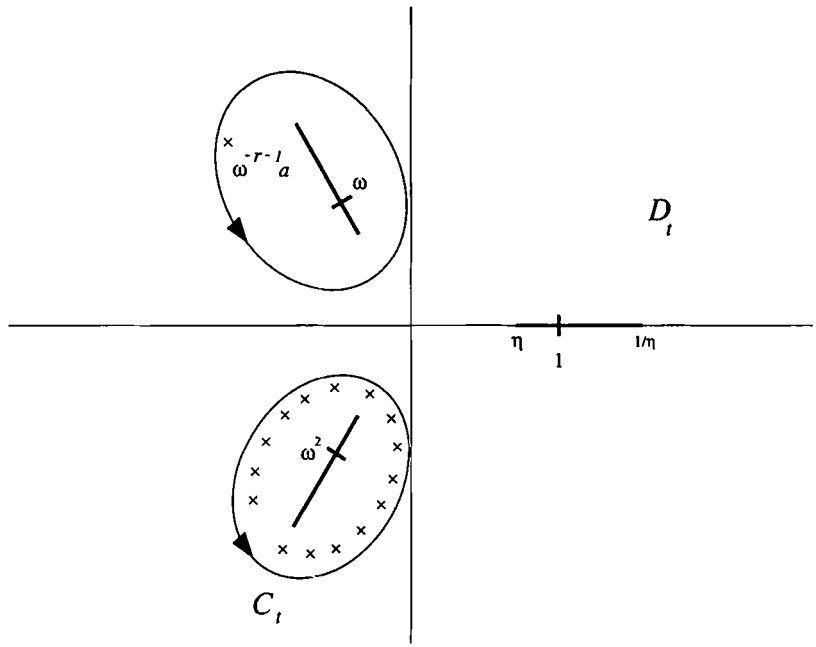

Fig. 2. The cut complex t_q plane, with $N=3$, with the branch cuts for λ_q as a function of t_q indicated by the bold lines. The zeros of $\tau_2(t_q)$ with $r=1$ are indicated ($\times$), with $L-1$ of them surrounding ω^2 and one at ωa. The contour $\mathscr{C}_t$, indicated, surrounds all the zeros of $\tau_2(t_q)$ except ωa. The domain $\mathscr{D}_t$ lies outside the two contours.

and relabeling the spins in the rows. In fact, from (2.20), in a diagonal representation we have the relation

$$(\hat{T}_q)_{\text{diag}} = D(T_q)_{\text{diag}} (f_{pq}/f_{p'q})^L \tag{2.23}$$

where D is some diagonal matrix which is independent of q.

We can choose the variables x_p, y_p, t_p, $x_{p'}$, $y_{p'}$, $t_{p'}$, x_q, y_q, t_q so that they all lie on the unit circle, and arrange them in the following order:

$$\begin{aligned}\arg(x_p), \arg(x_{p'}) \leqslant \arg(x_q) \leqslant \arg(y_p), \arg(y_{p'}) \\ \leqslant \arg(y_q) \leqslant \arg(\omega x_p), \arg(\omega x_{p'})\end{aligned} \tag{2.24}$$

and

$$\arg(t_p), \arg(t_{p'}) \leqslant \arg(t_q) \leqslant \arg(\omega t_p), \arg(\omega t_{p'}) \tag{2.25}$$

If we choose t_p, $t_{p'}$, and t_q to satisfy (2.25), then there is a unique choice of x_p, x_q, etc., that satisfies (2.24). If $-2\pi/N < \arg(t_q) < 0$, then we have $|\lambda_q| < 1$, and if $0 < \arg(t_q) < 2\pi/N$, then we have $|\lambda_q| > 1$, and similarly for t_p, λ_p and $t_{p'}$, $\lambda_{p'}$.

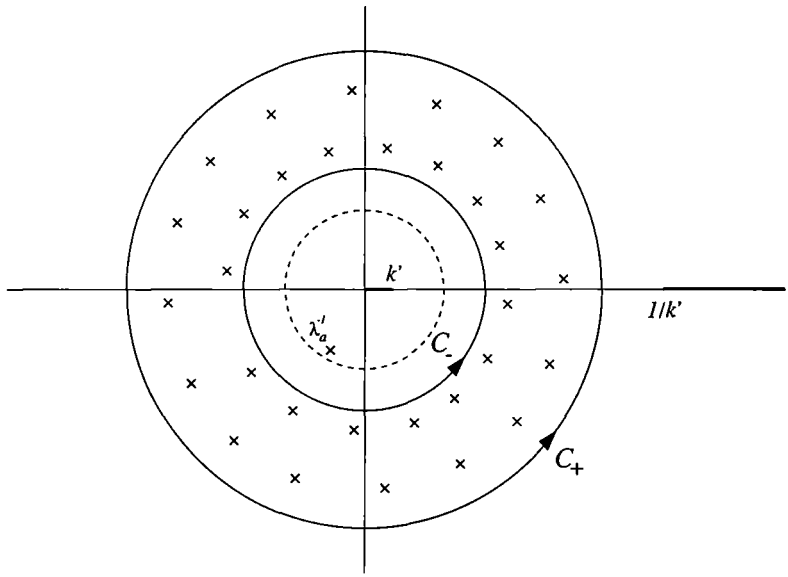

Fig. 3. The cut λ_q plane, showing the branch cuts for t_q as a function of λ_q. The zeros of the polynomial $\hat{S}(\lambda_q)$ are indicated (×), with $(N-1)L-r$ of them lying between the contours $\mathscr{C}_+$ and $\mathscr{C}_-$, and r lying inside the unit circle (the broken line), where we indicate the case $r=1$.

Once we have chosen the p, p', and q variables thus, the Boltzmann weights (2.9) and (2.10) are real and positive, so the model is physical.

The variables t_q and λ_q are related by (2.6), so that λ_q is a two-valued function of t_q, and t_q is an N-valued function of λ_q. The Riemann surface for λ_q as a function of t_q consists of two complex t_q planes (or "*sheets*") joined at the branch cuts shown in Fig. 2 (for $N=3$). The N branch cuts lie between the points $t_q=\omega^j\eta$ and $\omega^j\eta^{-1}$, where $j=0, 1,..., N-1$. When t_q is continued along the unit circle through any of the branch cuts, it goes from the sheet on which $|\lambda_q|>1$ to the sheet on which $|\lambda_q|<1$, or vice versa.

On the other hand, the Riemann surface for t_q as a function of λ_q consists of N sheets, joined at branch cuts between the points 0, k' and $1/k'$, ∞, which are shown in Fig. 3.

2.1. The Partition Function and Interfacial Tension

The interfacial tension is defined as follows. For nonskewed boundary conditions, $r=0$, the system has a ferromagnetically ordered ground state. This means that at sufficiently low temperatures (in particular, in the zero-temperature limit $k'\to 0$), one would expect to see the majority of the spins in the same state, σ say, where $0\leqslant\sigma\leqslant N-1$, the N possible ground states occurring with equal probability.

When $r=1$, the system should still have an ordered lowest-energy state in the limit $k'\to 0$, but the skewed boundary conditions are incompatible with all the spins being in the same state throughout the entire lattice. Down one vertical boundary, the spins are in state σ, while down the other, they are in state $\sigma-1$. In between, in the zero-temperature limit, there must be some line running down the lattice separating these two phases of the system; this interface may meander to the left or right, but the mean direction will be downward.

Still considering the limit $k'\to 0$, but with $r=2$, spins near one boundary of the lattice will be once more in some state σ, while those near the other will be in state $\sigma-2$. In between, there are *a priori* two possibilities.

One is that there will be a single interface separating the two phases of the system, with spins σ to the left and $\sigma-2$ to the right, and spins $\sigma-1$ not occurring. Otherwise, spins in state $\sigma-1$ could occur between the phases σ and $\sigma-2$, and so there would be two interfaces running down the lattice. In the latter case, the phase $\sigma-1$ is said to wet the σ, $\sigma-2$ phases.

The $r>2$ case is the obvious generalization of the $r=2$ case.

These interfaces should be well defined in the $k'\to 0$ limit, with different phases of the system containing spins of only the one value. As k' increases, the system will become more disordered, and the interfaces will

widen and begin to blur. The various phases will now no longer have all spins identical, but, for instance, the phase σ will still have the majority of its spins in the state σ. As $k' \to 1$, the system has a phase transition into a completely disordered state.

Associated with each of these interfaces is a surplus of energy needed to break the ground-state configuration. This energy is called the interfacial tension.

The partition function of the chiral Potts model with a skew parameter r on our $L \times M$ lattice is defined as

$$Z_r = \sum_{\{\sigma\}} \prod_{\langle i,j \rangle} W(\sigma_i - \sigma_j) \prod_{\langle i,k \rangle} \bar{W}(\sigma_i - \sigma_k) \tag{2.26}$$

where the sum is over all the values of all the spins σ, and the products are over all the SW $\to$ NE edges $\langle i, j \rangle$, and all SE $\to$ NW edges $\langle i, k \rangle$. In terms of the eigenvalues $T_q \hat{T}_q$, this is

$$Z_r = \sum (T_q \hat{T}_q)^M \tag{2.27}$$

where the sum is over all N^L eigenvalues.

For large L and M, the partition function will be of the form

$$Z_r \sim \exp[(-2LM\psi - M\varepsilon_r)/k_B \mathscr{T}] \tag{2.28}$$

where ψ, the free energy per site in the thermodynamic limit, is defined as

$$-\psi/k_B \mathscr{T} = \lim_{L,M \to \infty} (2LM)^{-1} \ln Z_r \tag{2.29}$$

The interfacial tension ε_r between phases σ and $\sigma - r$ is defined as

$$-\varepsilon_r/k_B \mathscr{T} = \lim_{L,M \to \infty} M^{-1} \ln(Z_r/Z_0) \tag{2.30}$$

where k_B is Boltzmann's constant and $\mathscr{T}$ is the temperature.

2.2. The Functional Relations

The functional relations for the model with skewed boundary conditions which are derived in ref. 1 are summarized here. Define the functions

$$z(t_q) = [\omega \mu_p \mu_{p'} (t_p - t_q)(t_{p'} - t_q)]^L \tag{2.31}$$

$$\alpha_q = [\lambda_q (y_p^N - x_q^N)(y_{p'}^N - x_q^N)/k']^L \tag{2.32}$$

$$\bar{\alpha}_q = [\lambda_q^{-1} (y_p^N - y_q^N)(y_{p'}^N - y_q^N)/k']^L \tag{2.33}$$

where (2.32) and (2.33) can be written as

$$\begin{aligned} \alpha_q &= [k'(1-\lambda_p\lambda_q)(1-\lambda_{p'}\lambda_q)/k^2\lambda_q]^L \\ \bar{\alpha}_q &= [k'(\lambda_q-\lambda_p)(\lambda_q-\lambda_{p'})/k^2\lambda_q]^L \end{aligned} \tag{2.34}$$

so if we write $\alpha_q = \alpha(\lambda_q)$, then $\bar{\alpha}_q = \alpha(1/\lambda_q)$.

The functional relations below define the set of matrices $\tau_j(t_q)$, $j=0,\ldots,$ $N+1$, and the transfer matrices T_q and $\hat{T}_q$. The τ_j matrices, whose entries are all polynomials in t_q of degree at most $(j-1)L$, are related by

$$\tau_0(t_q) = 0, \qquad \tau_1(t_q) = I \tag{2.35}$$

$$\tau_j(\omega t_q)\,\tau_2(t_q) = \omega^r Xz(\omega t_q)\,\tau_{j-1}(\omega^2 t_q) + \tau_{j+1}(t_q) \tag{2.36}$$

$$\tau_{N+1}(t_q) = \omega^r Xz(t_q)\,\tau_{N-1}(\omega t_q) + (\alpha_q + \bar{\alpha}_q)\,I \tag{2.37}$$

where I is the identity matrix, and X is the spin-shift operator, an N^L by N^L matrix with entries

$$X_{\sigma,\sigma'} = \prod_{j=1}^{L} \delta(\sigma_j, \sigma'_j + 1) \tag{2.38}$$

where the Kronecker delta functions are interpreted modulo N.

We have the following functional relation, relating the matrix $T_q = T(x_q, y_q, \mu_q)$ to the matrix $\tau_2(t_q)$:

$$\begin{aligned} \tau_2(t_q)\,T(x_q, \omega y_q, \mu_q) = \omega^r X &\left[\frac{\omega\mu_p\mu_{p'}(x_p - y_q)(t_{p'} - t_q)}{y_{p'} - y_q}\right]^L T(x_q, y_q, \mu_q) \\ &+ \left[\frac{(y_{p'} - \omega y_q)(t_p - \omega t_q)}{x_p - \omega y_q}\right]^L T(x_q, \omega^2 y_q, \mu_q) \end{aligned} \tag{2.39}$$

We use the following notation: for all complex numbers x and y and all integers m and n, let

$$(x, y)_{m,n} = \begin{cases} \prod_{j=m+1}^{n} (x - \omega^j y), & n \geqslant m \\ \prod_{j=n+1}^{m} (x - \omega^j y)^{-1}, & n \leqslant m \end{cases} \tag{2.40}$$

and define the following functions for all integers j, k, l such that $j = k + l$:

$$\Lambda_q^{(k,l)} = \left[\frac{(y_p, x_q)_{0,l-1}\,(y_q, x_p)_{-k,0}\,(y_q, y_{p'})_{0,N-k-1}}{N\mu_p^{-k}\mu_{p'}^{l}(x_{p'}, x_q)_{-1,l-1}}\right]^L \tag{2.41}$$

$$H_{pq}^{(j)} = [\omega^j \mu_p^j (t_p, t_q)_{-1,j-1}/(y_p^N - x_q^N)]^L \tag{2.42}$$

$$\bar{H}_{p'q}^{(j)} = [\mu_{p'}^{-j}(t_{p'}, t_q)_{j-1,N-1}/(x_{p'}^N - x_q^N)]^L \tag{2.43}$$

Define

$$\Xi_q^{(j)} = \omega^{rk} A_q^{(k,l)} X^k \hat{T}_{\bar{q}kl} \tag{2.44}$$

where $\hat{T}_{\bar{q}kl} = \hat{T}(\omega^k y_q, \omega^l x_q, \mu_q^{-1})$. It can be shown that $\Xi_q^{(j)}$ depends on k and l only via $j = k + l$. We then have the functional relation

$$T_q \Xi_q^{(j)} = \bar{H}_{p'q}^{(j)} \tau_j(t_q) + \omega^{jr} X^j H_{pq}^{(j)} \tau_{N-j}(\omega^j t_q) \tag{2.45}$$

Finally, we define the matrix $S(x_q, y_q)$ as follows:

$$S(x_q, y_q) = c(x_p^N - y_q^N)^{(N-1)L} \prod_{j=1}^{N-1} \left(\frac{y_{p'} - \omega^j y_q}{x_p - \omega^j y_q} \right)^{jL} \prod_{j=0}^{N-1} T(\omega^j x_q, y_q) \tag{2.46}$$

In ref. 7 it was shown that its entries are actually polynomials in λ_q of degree $(N-1)L$. For our calculations it is appropriate to define the related matrix $\hat{S}(\lambda_q) = \lambda_q^{(N-1)L} S(\lambda_q^{-1})$ as

$$\begin{aligned} \hat{S}(\lambda_q) = {} & c\lambda_q^{(N-1)L} (x_{p'}^N - x_q^N)^{(N-1)L} \\ & \times \prod_{j=1}^{N-1} \left(\frac{y_p - \omega^j x_q}{x_{p'} - \omega^j x_q} \right)^{jL} \prod_{j=0}^{N-1} \hat{T}(\omega^j y_q, x_q) \end{aligned} \tag{2.47}$$

where we choose the q-independent constant c to be

$$c = N^{-NL/2} (\lambda_p \lambda_{p'})^{(N-1)L/4} \tag{2.48}$$

If we can determine the matrices $\tau_j(t_q)$ and $\hat{S}(\lambda_q)$, then we can use Eq. (2.45) to calculate the transfer matrix T_q^N. For, (2.45) defines a set of N equations, where $j = 0, 1, \ldots, N-1$. If we take the product of these N equations, then the right-hand side will be a known function of t_q and λ_q, and the left-hand side will be proportional to $T_q^N \hat{S}(\lambda_q)$. Then $\hat{T}_q$ follows from (2.23), and we can compute the eigenvalues of $T_q \hat{T}_q$.

In Sections 3 and 4 we find integral expressions for the eigenvalues of the matrices $\tau_j(t_q)$ and $\hat{S}(\lambda_q)$ which are valid in appropriate domains of the t_q and λ_q planes, and we use these to calculate a band of L complex largest eigenvalues of $T_q \hat{T}_q$ in the large-L limit. From this band of eigenvalues we can determine the free energy and interfacial tension of the model.

2.3. The Solution

In refs. 6 and 7, the free energy of the chiral Potts model was calculated for $|\lambda_p|, |\lambda_q| < 1$ and $-2\pi/N < \arg(t_p), \arg(t_q) < 0$, and with $p = p'$. Define the functions $A(\lambda_p, t_q)$ and $B(\lambda_p, \lambda_q)$ as

$$A(\lambda_p, t_q) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}} \sum_{j=1}^{N-1} (N-j)\ln[\omega^{-j/2}\Delta(\theta) - \omega^{j/2} t_q]\, d\theta \tag{2.49}$$

$$B(\lambda_p, \lambda_q) = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}} \frac{1+\lambda_q e^{i\phi}}{1-\lambda_q e^{i\phi}} \times \sum_{j=1}^{N-1} (N-2j)\ln[\omega^{-j/2}\Delta(\theta) - \omega^{j/2}\Delta(\phi)]\, d\theta\, d\phi \tag{2.50}$$

where $\Delta(\theta)$ is the function

$$\Delta(\theta) = [(1+k'^2 - 2k'\cos\theta)/k^2]^{1/N} \tag{2.51}$$

where we choose $\Delta(\theta)$ to be real when θ is real.

With $r=0$ (periodic boundary conditions), there is a unique maximum eigenvalue, given by

$$N\ln(T_q \hat{T}_q) = 2L[N\ln g_{pq} + \tfrac{1}{2}(N-1)\ln(\lambda_q/\lambda_p) + A(\lambda_p, t_q) - A(\lambda_q, t_p) - B(\lambda_p, \lambda_q)] \tag{2.52}$$

for L large, and so the free energy per site ψ is given by

$$-N\psi/k_B\mathcal{T} = N\ln g_{pq} + \tfrac{1}{2}(N-1)\ln(\lambda_q/\lambda_p) + A(\lambda_p, t_q) - A(\lambda_q, t_p) - B(\lambda_p, \lambda_q) \tag{2.53}$$

[Note that the functions $A_{pq} = A(\lambda_p, t_q)$ and $B_{pq} = B(\lambda_p, \lambda_q)$ differ slightly from their definitions in ref. 7.]

Following ref. 1, we will be considering the region $|\lambda_q| > 1$, $0 < \arg(t_q) < 2\pi/N$. The model here is physical also, and the maximum eigenvalue should be given by the analytical continuation of (2.52). By writing Eqs. (2.49) and (2.50) as integrals around the unit circle, and deforming the contours of integration as λ_q crosses the unit circle, we find that $A(\lambda_q, t_p)$ and $B(\lambda_p, \lambda_q)$ have the following analytical continuation formulas for $|\lambda_q| > 1$;

$$A_{ac}(\lambda_q, t_p) = -A(1/\lambda_q, t_p) + 2\sum_{j=1}^{N-1} (N-j)\ln(\omega^{-j/2} t_q - \omega^{j/2} t_p) \tag{2.54}$$

$$B_{ac}(\lambda_p, \lambda_q) = -B(\lambda_p, 1/\lambda_q) + A(\lambda_p, t_q) - C(\lambda_p, t_q) \tag{2.55}$$

where we define the function $C(\lambda_p, t_q)$ as

$$C(\lambda_p, t_q) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}} \sum_{j=1}^{N-1} j\ln[\omega^{-j/2}\Delta(\theta) - \omega^{j/2} t_q]\, d\theta \tag{2.56}$$

When $|\lambda_q| > 1$, the eigenvalue $T_q \hat{T}_q$ is therefore given by (2.52), but with the integrals A and B replaced by their analytic continuations here defined; so we have for $|\lambda_q| > 1$

$$N \ln(T_q \hat{T}_q) = 2LE_{pq} \tag{2.57}$$

where we define

$$E_{pq} = N \ln g_{pq} + \tfrac{1}{2}(N-1) \ln(\lambda_q/\lambda_p) + A(\lambda_q^{-1}, t_p) + C(\lambda_p, t_q) + B(\lambda_p, \lambda_q^{-1}) - 2 \sum_{j=1}^{N-1} (N-j) \ln[\omega^{-j/2} t_q - \omega^{j/2} t_p] \tag{2.58}$$

and so the free energy is

$$-N\psi/k_B \mathscr{T} = E_{pq} \tag{2.59}$$

The interfacial tension ε_r was found in ref. 2 for the superintegrable chiral Potts model, where it was argued that it should be the same as that of the general physical model. We define the function

$$v_r = r \ln \mu_q - \ln A(\omega^{-1/2} m t_q) + \frac{1}{2} \ln h(\eta m) + m \int_\eta^{1/\eta} \frac{h'(my)}{\pi h(my)} \Psi(\lambda_q, y)\, dy \tag{2.60}$$

where m is some complex number, and the functions $A(t)$, $h(t)$, and $\Psi(\lambda, t)$ are, respectively,

$$A(t) = \prod_{j=1}^{r} (1 + \omega^{j-(r+1)/2} t) \tag{2.61}$$

$$h(t) = A(\omega^{-1/2} t)\, A(\omega^{1/2} t) \tag{2.62}$$

$$\Psi(\lambda, t) = \tan^{-1}\left[\frac{\lambda - 1}{\lambda + 1} \left(\frac{\eta^{-N} - t^N}{t^N - \eta^N} \right)^{1/2} \right] \tag{2.63}$$

when $\eta < t < \eta^{-1}$. The interfacial tension was found in ref. 2 to be

$$\varepsilon_r / k_B \mathscr{T} = 2(v_r)_{\text{saddle}} \tag{2.64}$$

where by $(v_r)_{\text{saddle}}$ we mean (2.60) evaluated at its saddle point in the complex m plane.

3. THE LOW-TEMPERATURE LIMIT, $k' \to 0$

In order to solve the functional relations of the last section, we first consider the low-temperature $k' \to 0$ limit of ref. 1. In taking the limit

$k' \to 0$, we can choose the solutions of (2.4) and (2.5) so that x_p, y_p, $x_{p'}$, $y_{p'}$, $x_q \to 1$, while y_q, μ_p, and $\mu_{p'}$ are held fixed. This implies that $t_q = y_q$, where $0 < \arg(t_q) < 2\pi/N$; also $|\lambda_q| > 1$ and $|\lambda_p|$, $|\lambda_{p'}| < 1$. It is shown in ref. 1 that in this limit the matrix T_q simplifies, so that we can write

$$T_q = \mu_q^{-r} F(t_q) \tag{3.1}$$

where $F(t_q)$ is a matrix whose entries are polynomials in t_q of degree at most r. If we let $F(t_q)$ also denote an eigenvalue of this matrix, then (2.39) becomes

$$\tau_2(t_q)\, F(\omega t_q) = \omega^{r+Q}[\omega \mu_p \mu_{p'}(1-t_q)]^L\, F(t_q) + (1-\omega t_q)^L\, F(\omega^2 t_q) \tag{3.2}$$

The zeros of $F(t_q)$ were calculated in ref. 1 as the solutions to a set of Bethe-ansatz type equations. Let the zeros of $F(t_q)$ be $t_q = e^{2i\alpha_j}$ for $j = 1,..., r$, so

$$F(t_q) \propto \prod_{j=1}^{r} (1 - e^{-2i\alpha_j} t_q) \tag{3.3}$$

where the proportionality constant is independent of t_q.

In ref. 1, Baxter notes that there are two distinct types of eigenvector for $r \geqslant 2$. The eigenvectors are either combinations of "plane waves" or "bound states," and Baxter notes that the fully bound states give the greater contribution to the partition function. Letting $\varepsilon = |\mu_p \mu_{p'}|$, then for L large, these satisfy

$$|\varepsilon^r \sin(\alpha_1 - r\pi/N)/\sin \alpha_1| = 1, \qquad \alpha_j = \alpha_1 + (1-j)\,\pi/N \tag{3.4}$$

We can gain some insight into the location of the zeros of $F(t_q)$ and $\tau_2(t_q)$ by considering the limit $\varepsilon \to 0$ and then varying ε to see how the location of the zeros depends on ε. In order to maintain the first equation of (3.4) as $\varepsilon \to 0$, we choose $\alpha_1 = n\pi$, where n is an arbitrary integer. Then from the second equation of (3.4), we have $\alpha_j = -n\pi + (1-j)\,\pi/N$, and as n is arbitrary, we can choose $\alpha_j = (1-j)\,\pi/N$, $j = 1,..., r$, for the bound-state solutions. With this choice, the polynomial $F(t_q)$ is given by

$$F(t_q) = \prod_{j=1}^{r} (1 - \omega^{j-1} t_q) \tag{3.5}$$

This is a polynomial of degree r which has simple zeros lying exactly at the roots of unity $1, \omega^{-1},..., \omega^{1-r}$.

As $\varepsilon = 0$, the first term of the right-hand side of (3.2) vanishes, so $\tau_2(t_q)$ is given by

$$\tau_2(t_q) = (1 - \omega t_q)^{L-1} (1 - \omega^{r+1} t_q) \tag{3.6}$$

This is a polynomial of degree L, which has a zero of multiplicity $L-1$ at ω^{-1}, and a simple zero at ω^{-r-1}. For $r=0$, $\tau_2(t_q) = (1-\omega t_q)^L$, so we see that a nonzero skew parameter shifts one of the zeros of $\tau_2(t_q)$ from ω^{-1} to another root of unity.

Similarly, we calculate the other $\tau_j(t_q)$ functions in the $\varepsilon \to 0$ limit. From (2.31), $z(t_q) \to 0$, so Eqs. (2.35)–(2.37) imply

$$\tau_j(t_q) = \prod_{k=1}^{j-1} \tau_2(\omega^{k-1} t_q) = \prod_{k=1}^{j-1} (1 - \omega^k t_q)^{L-1} (1 - \omega^{r+k} t_q) \tag{3.7}$$

Once again, the introduction of the skew parameter r merely shifts some of the zeros of the polynomials $\tau_j(t_q)$.

We then increase ε to a small but nonzero value (still considering the limit $k' \to 0$). All of the zeros of the polynomials $F(t_q)$ and $\tau_j(t_q)$ will move out from the roots of unity as ε increases, but only by a small amount. In fact, we find that they lie on circles centered at roots of unity, with radii that are proportional to positive powers of ε. Hence (3.3) becomes

$$F(t_q) = \prod_{j=1}^{r} (a_j - \omega^{j-1} t_q) \tag{3.8}$$

where the a_j are given up to constants of order unity by

$$1 - a_1 = O(\varepsilon^r), \qquad a_j = a_1 + O(\varepsilon^{(r-j+1)L}) \quad \text{for } j = 2, \ldots, r \tag{3.9}$$

(There are in fact L possible choices for a_1, spaced around a circle with a radius that is proportional to ε^r. The particular choice of a_1 determines the other a_j, and different choices of a_1 define different bound-state eigenvalues.) In particular, for ε nonzero, we find that $\tau_2(t_q)$ has a circle of $L-1$ zeros surrounding ω^{-1}, with a radius proportional to ε when $r=1$ and $\varepsilon^{L/(L-1)}$ otherwise, and a single outlier which is a distance proportional to ε^r from ω^{-r-1}.

3.1. Generalization to Arbitrary k'

We can use this information to work out integral formulas for the polynomials $\tau_2(t_q)$, $\tau_3(t_q)$,..., $\tau_N(t_q)$ in the large-lattice $L \to \infty$ limit. We will use Cauchy's integral formula to write expressions for the polynomials by surrounding their zeros by contours, and obtain integral expressions that are valid outside the contour.

We have located the zeros in the limit $k' \to 0$, and for ε small but nonzero. We wish to vary k' also, so our results will be useful at arbitrary temperatures, and in particular valid as the system approaches criticality.

The zeros of $\tau_2(t_q)$ may move further as k' increases, and the ring they lie on will surround the branch cut for λ_q as a function of t_q. The zeros of $\tau_3(t_q),\ldots, \tau_N(t_q)$ will move likewise as we increase k', but for k' small enough, will still lie in a neighborhood of the roots of unity around which they originally lay.

When $r=0$, we see that the zeros of these polynomials will be clustered around all the roots of unity except $t_q=1$. Thus there will exist a domain in the complex t_q plane which excludes all the zeros of the functions $\tau_2(t_q)_{r=0}$, $\tau_3(t_q)_{r=0},\ldots,$ $\tau_N(t_q)_{r=0}$, but which will contain a neighborhood of $t_q=1$. We call this domain $\mathscr{D}_t$. Thus $\tau_j(t_q)$ can have at most a simple zero in $\mathscr{D}_t$. This domain also excludes the branch cuts for λ_q as a function of t_q, apart from the one between η and $1/\eta$.

3.2. The Polynomial $\tau_2(t_q)$

We follow the method of refs. 6 and 7. From Eqs. (2.36) and (2.37), we see that

$$\tau_2(t_q)\,\tau_2(\omega t_q)\cdots\tau_2(\omega^{N-1}t_q)=\alpha_q+\bar{\alpha}_q+\xi \tag{3.10}$$

where ξ is a sum of products of the polynomials τ_2 and z. The left-hand side of (3.10) is a polynomial in t_q^N of degree L, which we will call $M(t_q)$. Let its zeros be at $a_1^N,\ldots, a_L^N$, so

$$M(t_q) \propto \prod_{j=1}^{L} (a_j^N - t_q^N) \tag{3.11}$$

(In the limits $k', \varepsilon \to 0$, we found $a_j=1, j=1,\ldots, L$; for k' and ε different from zero, we expect the a_j to lie in some neighborhood of unity.) Following our comments regarding the location of the zeros in the $k' \to 0$ limit, we can write

$$\tau_2(t_q) \propto (a_1-\omega^{r+1}t_q)\prod_{j=2}^{L}(a_j-\omega t_q) \tag{3.12}$$

The zeros of $M(t_q)$ will occur in N sets of L, clustered around the roots of unity $1, \omega,\ldots, \omega^{N-1}$. Let $\mathscr{C}_t$ be a simple closed contour oriented in the positive direction which surrounds all the zeros of $M(t_q)$ that lie near the point ω^{-1}, i.e., $\mathscr{C}_t$ just surrounds the hole in $\mathscr{D}_t$ around ω^{-1}. Then inside $\mathscr{C}_t$ lie the $L-1$ zeros of $\tau_2(t_q)$ that are made up of the L original

zeros of the polynomial, less the one that was shifted; and the shifted zero of $\tau_2(\omega^{-r}t_q)$.

Let $a=a_1$, so $\omega^{-1}a$ is the zero of $\tau_2(t_q)$ which gets shifted when the boundary conditions are skewed. Then using Cauchy's integral formula and Eq. (3.12), then integrating, we can write

$$\ln\left\{\tau_2(s)\frac{a-\omega s}{a-\omega^{r+1}s}\right\}=\frac{1}{2\pi i}\oint_{\mathscr{C}_t}\ln C(t-s)\frac{d}{dt}\ln M(t)\,dt \tag{3.13}$$

where C is a constant of integration, to be determined. This form of $\tau_2(s)$ is valid whenever s is outside $\mathscr{C}_t$.

Consider the limit $L\to\infty$. From the low-temperature limit, when k' and $\varepsilon\to 0$ with $k'\ll\varepsilon^{N/2}$, we note the following:

$$\bar{\alpha}_q=O(1),\qquad \alpha_q=O(\varepsilon^{NL}),\qquad z(t_q)=O(\varepsilon^{L}),\qquad \tau_2(t_q)\leqslant O(1) \tag{3.14}$$

With this in mind, the dominant term in (3.10) is $\bar{\alpha}_q$, all the other terms decaying exponentially to zero as $L\to\infty$. If we assume that this behavior holds for general k' and ε, then in the $L\to\infty$ limit, we may replace $M(t_q)$ with $\bar{\alpha}_q$. Substituting this into (3.13), we have the exact expression

$$\ln\left\{\tau_2(s)\frac{a-\omega s}{a-\omega^{r+1}s}\right\}=\frac{1}{2\pi i}\oint_{\mathscr{C}_t}\ln C(s-t)\frac{d}{dt}\ln\bar{\alpha}(\lambda)\,dt \tag{3.15}$$

for L large and s outside $\mathscr{C}_t$. The λ occurring in the integrand is chosen so that $|\lambda|>1$, to be consistent with the low-temperature-limit calculations. This integral is taken around the contour of integration $\mathscr{C}_t$ on the $|\lambda_q|>1$ sheet of the t_q Riemann surface. However, the integrand of (3.15) is now analytic inside the part of the cut t_q plane that lies inside $\mathscr{C}_t$ [the function $\bar{\alpha}(\lambda)$ has zeros when $\lambda=\lambda_p$ or $\lambda_{p'}$, but as $|\lambda_p|$, $|\lambda_{p'}|<1$, the zeros of $\bar{\alpha}(\lambda)$ lie on the other t_q plane]. We can thus shrink the contour of integration down until it just surrounds the branch cut from $\omega^{-1}\eta$ to $\omega^{-1}\eta^{-1}$.

As t goes around this branch cut in the t plane, λ goes once around the unit circle in the λ plane in the positive direction. Changing variables from t to λ in the integral, we obtain

$$\ln\left\{\tau_2(s)\frac{a-\omega s}{a-\omega^{r+1}s}\right\}=\frac{1}{2\pi i}\oint_{\mathscr{C}_\lambda}\ln C(s-t)\frac{d}{d\lambda}\ln\bar{\alpha}(\lambda)\,d\lambda \tag{3.16}$$

where $\mathscr{C}_\lambda$ represents the unit circle in the λ plane.

The constant C is evaluated from the $L \to \infty$ limit of (3.10). We then make the substitution $\lambda = e^{i\theta}$, so $t = \omega^{-1}\Delta(\theta)$. Replacing s by t_q, we have

$$\ln\left\{\tau_2(t_q)\frac{a-\omega t_q}{a-\omega^{r+1}t_q}\right\} = \frac{L}{4\pi}\int_0^{2\pi}\left(\frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}}+\frac{1+\lambda_{p'}e^{i\theta}}{1-\lambda_{p'}e^{i\theta}}\right)\ln[\Delta(\theta)-\omega t_q]\,d\theta \tag{3.17}$$

as our integral formula for $\tau_2(t_q)$, which we can write as

$$\tau_2(t_q) = \frac{a-\omega^{r+1}t_q}{a-\omega t_q}\,\tau_2(t_q)_{r=0} \tag{3.18}$$

This expression is exact subject to our assumptions about the location of the zeros, for t_q outside $\mathscr{C}_t$ and L large.

3.3. The Polynomials $\tau_j(t_q)$

We can use Eqs. (2.35)–(2.37) together with (3.18) to derive an expression for the functions $\tau_3(t_q),\ldots,\tau_N(t_q)$ in the limit $L \to \infty$. Iterating (2.35)–(2.37), we have

$$\tau_j(t_q) = \tau_2(t_q)\,\tau_2(\omega t_q)\cdots\tau_2(\omega^{j-2}t_q) + \xi' \tag{3.19}$$

where ξ' is a function made up of sums of products of τ_2 and z functions as before. As long as t_q is not a zero of any of the τ_2 functions appearing in this formula (i.e., for t_q in $\mathscr{D}_t$), then the z functions will be exponentially smaller than any of the τ_2 functions, and therefore negligible as $L \to \infty$. Therefore, we have

$$\tau_j(t_q) = \prod_{k=1}^{j-1}\tau_2(\omega^{k-1}t_q) \tag{3.20}$$

which is valid for L large and for $t_q \in \mathscr{D}_t$.

3.4. Possible Values of a

We noted that there were L values a could take in the $k' \to 0$ limit, and each one corresponded to a different bound-state eigenvalue of the transfer matrix. For general k' we derive an equation for the possible values of a as follows. Consider the $j = N - r$ case of Eq. (2.45). In the $k' \to 0$ limit we see from (3.1) and (3.8) that T_q has a zero when $t_q = a$, and we see later that

this is true for k' nonzero also (though we do not assume it here). From (3.20) and (3.18), remembering that when $r=0$ the zeros of $\tau_2(t_q)$ are clustered around ω^{-1}, we can see that the polynomials $\tau_r(\omega^{N-r}t_q)$ and $\tau_{N-r}(t_q)$ are both nonzero at $t_q=a$. If we then substitute $t_q=a$ into (2.45), the relation becomes

$$\bar{H}^{(N-r)}_{p'q}/H^{(N-r)}_{pq} = -\omega^{jr+Q}\tau_r(\omega^{N-r}a)/\tau_{N-r}(a) \tag{3.21}$$

which can be written using (3.20) and (3.17) as

$$\begin{aligned} L \ln\left[(\omega\mu_p\mu_{p'})^r \prod_{j=1}^{r}(t_p-\omega^{-j}a)(t_{p'}-\omega^{-j}a)\right] \\ = Q\ln\omega + \frac{L}{4\pi}\int_0^{2\pi}\left(\frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}}+\frac{1+\lambda_{p'}e^{i\theta}}{1-\lambda_{p'}e^{i\theta}}\right) \\ \times \sum_{j=1}^{r}\ln\{[\Delta(\theta)-\omega^{-j}a][\Delta(\theta)-\omega^{-j+1}a]\}\,d\theta \end{aligned} \tag{3.22}$$

This integral equation is exact in the limit $L\to\infty$, and for finite L it is exact up to terms that vanish exponentially with L. We have plotted the various allowed values of a in Fig. 4, numerically solving the equation for $L=60$ and $k'=0.104$, $t_p=11/20$, $\mu_p=1/2$. Note that there are L possible values for a, that they lie on a closed contour surrounding the branch cut for λ_q as a function of t_q, and that they are distributed nonuniformly around this contour. As $L\to\infty$, the solutions to (3.22) become densely spaced on this contour.

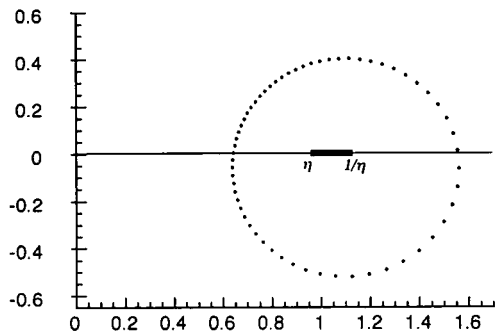

Fig. 4. Solutions to Eq. (3.22) for $N=3$, $r=1$, $k'=0.104$, $\lambda_p=1/8$, $t_p=11/20$, and $L=60$. The branch cut for λ_q as a function of t_q is shown in bold, between the branch points η and $1/\eta$.

4. THE POLYNOMIAL $\hat{S}(\lambda_q)$

4.1. The $k' \to 0$ Limit Again

We now derive an expression for the polynomial $\hat{S}(\lambda_q)$, which was defined in Eq. (2.47). Call the RHS of (2.45) $r_j(t_q, \lambda_q)$, $j = 0, 1, \ldots, N-1$; as the polynomials $\tau_j(t_q)$ and $\tau_{N-j}(\omega^j t_q)$ for $j = 0, 1, \ldots, N-1$ are exactly known functions of t_q for $t_q \in \mathscr{D}_t$, so the function $r_j(t_q, \lambda_q)$ is an exactly known function of λ_q and t_q for $t_q \in \mathscr{D}_t$. The zeros of $r_j(t_q, \lambda_q)$ are zeros of any of the functions on the LHS of (2.45), i.e., zeros of either T_q, $\hat{T}_{\bar{q}j0}$, or $\Lambda_q^{(j,0)}$.

We need to locate the zeros of $\hat{S}(\lambda_q)$, and as per the last section, we start by considering the limit $k' \to 0$. Then (2.41)–(2.43) become

$$\Lambda_q^{(j,0)} \propto (\lambda_q \mu_p^{j-N})^L, \qquad H_{pq}^{(j)} \propto (\lambda_q \mu_p^{j-N})^L, \qquad \bar{H}_{p'q}^{(j)} \propto (\mu_{p'}^{N-j}/k')^L \tag{4.1}$$

where all the proportionality constants are of order unity. Also, in this limit T_q is given by (3.1), and so if we can locate the zeros of $r_j(t_q, \lambda_q)$ in the λ_q plane, we can identify which belong to $\hat{T}_{\bar{q}j0}$ and hence to $\hat{S}(\lambda_q)$.

We first consider briefly what happens when $r = 0$ (the case considered in refs. 6 and 7). From (3.1), T_q is order unity, so all of the zeros of r_j will be zeros of $\hat{T}_{\bar{q}j0}$. The τ_j polynomials are order unity also, so the functional relation simplifies, becoming, for $j = 1, \ldots, N-1$,

$$r_j \propto c_1 \varepsilon^{(N-j)L} \lambda_q^{-L} + c_2 k'^L \tag{4.2}$$

where c_1 and c_2 are order unity. The zeros of the functions r_j for $j = 1, \ldots, N-1$ lie on circles centered at the origin and with radii proportional to ε^{N-j}/k', while r_0 is order unity. All $(N-1)L$ of these zeros are contained inside the annulus $1 < |\lambda_q| < 1/k'$, and these make up all the zeros of $\hat{S}(\lambda_q)$.

Guided by the results of the last section, we expect that some of the zeros of $\hat{S}(\lambda_q)$ will shift when we have r nonzero. From Eqs. (3.1), (3.8), and (3.9), we see that when $t_q \in \mathscr{D}_t$, then $T_q \propto (\varepsilon^r - k'\lambda_q)$, with a proportionality constant which is order unity. Also, $\tau_j(t_q)$ and $\tau_{N-j}(\omega^j t_q)$ are order unity unless $0 \leqslant j \leqslant N-r-1$ when $\tau_{N-j}(\omega^j t_q) \propto (\varepsilon^r - k'\lambda_q)$, or $N-r+1 \leqslant j \leqslant N-1$ when $\tau_j(t_q) \propto (\varepsilon^r - k'\lambda_q)$. The functional relation (2.45) becomes

$$r_j(t_q, \lambda_q) \propto c_1 \varepsilon^{(N-j)L} \tau_j(t_q)\, \lambda_q^{-L} + c_2 k'^L \tau_{N-j}(\omega^j t_q) \tag{4.3}$$

where c_1 and c_2 are order unity, and $\tau_j(t_q)$ or $\tau_{N-j}(\omega^j t_q)$ is either order unity or proportional to $(\varepsilon^r - k'\lambda_q)$, depending on the value of j. There are N such equations, with $j = 0, 1, \ldots, N-1$.

The zeros of r_j and $\hat{T}_{\bar{q}j0}$ are as follows; when $j=0$, then $r_0 \propto (\varepsilon^r - k'\lambda_q)$, and so r_0 has only a single zero, which in fact corresponds to the zero of T_q. Thus $\hat{T}_{\bar{q}00}$ is order unity.

When $1 \leqslant j \leqslant N-r-1$, then r_j has L zeros lying on circles centered at the origin and with radii proportional to $\varepsilon^{N-j-r/L}/k'$, and one outlier at a distance proportional to ε^r/k'. This outlier must belong to T_q, the L zeros lying on each of the circles belonging to $\hat{T}_{\bar{q}j0}$.

When $j=N-r$, the L zeros of r_j lie on a circle centered at the origin and with radius proportional to ε^r/k'. One of these must belong to T_q, and so the zeros of $\hat{T}_{\bar{q}(N-r,0)}$ are the remaining $L-1$ zeros that lie on the circle.

Finally, when $N-r+1 \leqslant j \leqslant N-1$, the zeros of r_j consist of $L-1$ points lying on circles centered at the origin and with radii proportional to $\varepsilon^{(N-j)L/(L-1)}/k'$ and one outlier at a distance proportional to ε^r/k'. The outlier belongs to T_q, the $L-1$ zeros spaced around each of the circles to $\hat{T}_{\bar{q}j0}$.

All the zeros of $\hat{T}_{\bar{q}j0}$, $j=1, 2,..., N-1$, are zeros of $\hat{S}(\lambda_q)$, and thus we have located $(N-1)L-r$ zeros of $\hat{S}(\lambda_q)$, all of which lie outside the unit circle, in the annulus bounded by the unit circle and the circle $|\lambda_q| = 1/k'$.

Let $\hat{S}_j(\lambda_q)$, $j=1, 2,..., N-1$, be the polynomial in λ_q which has the same zeros as $r_j(t_q, \lambda_q)$ when $t_q \in \mathscr{D}_t$. (The $j=0$ polynomial is simply a constant.) Then each $\hat{S}_j(\lambda_q)$ is a polynomial of degree $L+1$ if $j=1,...,$ $N-r-1$, and degree L if $j=N-r,..., N-1$. They all contain the factor $(\lambda_q - \lambda_a)$, where λ_a is the value of λ_q corresponding to $t_q = a$, with $|\lambda_a| > 1$. [This zero belongs to T_q in each case, and so is not a zero of $\hat{S}(\lambda_q)$.] Otherwise all of their zeros lie on circles centered at the origin and which lie inside the annulus $1 < |\lambda_q| < 1/k'$, and $\hat{S}(\lambda_q)$ contains the factor

$$(\lambda_q - \lambda_a)^{1-N} \prod_{j=1}^{N-1} \hat{S}_j(\lambda_q) \tag{4.4}$$

From Section 2, the polynomial $\hat{S}(\lambda_q)$ must have $(N-1)L$ zeros, so there are r zeros still to locate. We have only found the zeros that lie outside the unit circle, so there must be r zeros inside the unit circle. Consider the polynomial $S(\lambda_q)$ defined by (2.46). The product of the $T(\omega^j x_q, y_q)$ functions becomes a product of $F(\omega^j t_q)$ functions in the $k' \to 0$ limit, so $S(\lambda_q) \propto (a_1 - t_q)(a_2 - t_q) \cdots (a_r - t_q)$, when $t_q \in \mathscr{D}_t$. Denote the corresponding zeros in the λ_q plane as $\lambda_{a_1}, \lambda_{a_2},..., \lambda_{a_r}$, where each of the λ_{a_j}, $j=1,..., r$, are chosen so that $|\lambda_{a_j}| > 1$, so that $\hat{S}(\lambda_q)$ also has the factor

$$\prod_{j=1}^{r} (\lambda_q - \lambda_{a_j}^{-1}) \tag{4.5}$$

Hence in the $k' \to 0$ limit, we have located all $(N-1)\,L$ zeros of $\hat{S}(\lambda_q)$. We have

$$\hat{S}(\lambda_q) \propto \frac{(\lambda_q - \lambda_{a_1}^{-1}) \cdots (\lambda_q - \lambda_{a_r}^{-1})}{(\lambda_q - \lambda_a)^{N-1}} \prod_{j=1}^{N-1} \hat{S}_j(\lambda_q) \tag{4.6}$$

where the proportionality constant is order unity.

The distribution of the zeros of $\hat{S}(\lambda_q)$ is largely the same as the $r=0$ case, and we see again that when r is nonzero, r of the zeros of $\hat{S}(\lambda_q)$ move inside the unit circle.

4.2. The Polynomial $\hat{S}(\lambda_q)$ for k' Nonzero

We can now work out an integral equation for the polynomial $\hat{S}(\lambda_q)$, when k' is nonzero and L large. As $L \to \infty$, then from (3.9), we see that $a_j \to a$, $j=1,\ldots,r$, and so $\lambda_{a_j} \to \lambda_a$. Hence for L large, $\hat{S}(\lambda_q)$ has a zero of multiplicity r corresponding to the factor $(\lambda_q - \lambda_a^{-1})^r$.

For k' nonzero, we expect that the zeros of $r_j(\lambda_q)$ will still lie in largely the same distribution as they do in the $k' \to 0$ limit, but that perhaps they will shift as k' increases. We still expect $(N-1)\,L - r$ of the zeros to lie inside the annulus $1 < |\lambda_q| < 1/k'$, and that the r-fold zero at λ_a^{-1} will lie inside the region $k' < |\lambda_q| < 1$. The zeros outside the unit circle can be surrounded by two contours, one lying just outside the unit circle, called $\mathscr{C}_-$, and one lying outside $\mathscr{C}_-$, called $\mathscr{C}_+$, as indicated in Fig. 3. Both contours are contained inside the annulus $1 < |\lambda_q| < 1/k'$, and are oriented in the positive direction.

Using Cauchy's integral formula as in the previous section, we have

$$\frac{d}{d\lambda} \ln \hat{S}_j(\lambda) = \frac{1}{2\pi i} \oint_{\mathscr{C}_+} \frac{d\lambda'}{\lambda - \lambda'} \frac{d}{d\lambda'} \ln r_j(\lambda') - \frac{1}{2\pi i} \oint_{\mathscr{C}_-} \frac{d\lambda'}{\lambda - \lambda'} \frac{d}{d\lambda'} \ln r_j(\lambda') \tag{4.7}$$

$j=1,\ldots,N-1$, which is valid when either λ is inside $\mathscr{C}_-$ or outside $\mathscr{C}_+$. This expression is exact for finite L, and we can make it more explicit in the limit $L \to \infty$. From the $k' \to 0$ limit, we can see that on $\mathscr{C}_+$ the second term of (2.45) dominates, while on $\mathscr{C}_-$, the first does. Define the polynomial $\xi(\lambda)$ by

$$\xi(\lambda) = \prod_{j=1}^{N-1} \hat{S}_j(\lambda) \tag{4.8}$$

Then integrating with respect to λ, we have for λ outside $\mathscr{C}_+$ and $t' \in \mathscr{D}_t$

$$\ln \xi(\lambda) = c_1 + \frac{1}{2\pi i}\oint_{\mathscr{C}_+} d\lambda' \ln(\lambda - \lambda') \frac{d}{d\lambda'} \ln \left[\prod_{j=1}^{N-1} H_{pq}^{(j)} \tau_{N-j}(\omega^j t') \right]$$
$$- \frac{1}{2\pi i}\oint_{\mathscr{C}_-} d\lambda' \ln(\lambda - \lambda') \frac{d}{d\lambda'} \ln \left[\prod_{j=1}^{N-1} \bar{H}_{p'q}^{(j)} \tau_j(t') \right] \tag{4.9}$$

where c_1 is a q-independent constant of integration.

As the functions $H_{pq}^{(j)}$ and $\bar{H}_{pq}^{(j)}$, given by (2.42) and (2.43), respectively, only appear as logarithmic derivatives in the integrands, we can neglect any factors they possess that are independent of the q-variables (as these do not contribute to the integrals), so we have

$$\prod_{j=1}^{N-1} H_{pq}^{(j)} \propto (\lambda' - \lambda_p)^{L(N-1)} \prod_{j=1}^{N-1} (t_p - \omega^j t')^{-jL}$$
$$\prod_{j=1}^{N-1} \bar{H}_{p'q}^{(j)} \propto [\lambda'/(\lambda' - \lambda_{p'})]^{(N-1)L} \prod_{j=1}^{N-1} (t_{p'} - \omega^j t')^{jL} \tag{4.10}$$

Substituting these and (4.9) into (4.8) and expressing the integrands in terms of t' and λ' only (rather than x', y', etc.), and evaluating the integrals that contain only λ', we get

$$\ln \xi(\lambda) = c_2 + (N-1) \ln \bar{\alpha}_q$$
$$+ \frac{1}{2\pi i}\oint_{\mathscr{C}_+} d\lambda' \ln(\lambda - \lambda') \frac{d}{d\lambda'} \ln \left[\prod_{j=1}^{N-1} \frac{\tau_{N-j}(\omega^j t')}{(t_p - \omega^j t')^{jL}} \right]$$
$$- \frac{1}{2\pi i}\oint_{\mathscr{C}_-} d\lambda' \ln(\lambda - \lambda') \frac{d}{d\lambda'} \ln \left[\prod_{j=1}^{N-1} \tau_j(t')(t_{p'} - \omega^j t')^{jL} \right] \tag{4.11}$$

(where c_2 is a new q-independent constant of integration). The next step, following refs. 6 and 7, would be to integrate this expression by parts. However, the integrands have branch points when the factor $(a - t)$ occurs in the argument of the logarithm in the integrands, corresponding to branch points at 0, λ_a^{-1}, λ_a, and ∞ in the λ_q plane. We use (3.18) and (3.20) to manifest the factors $(a - t)$ explicitly, writing them as $a^N - t^N$, which is an analytic function of λ. Once these factors are removed, the functions remaining in the integrands contain no branch points, so the integrals are then single-valued around their respective contours, and hence we can perform an integration by parts. In fact, the resulting integrands have no zeros at all between the contours $\mathscr{C}_+$ and $\mathscr{C}_-$, so both of these contours can be shifted to the unit circle and the integrals combined to give

$$\ln \xi(\lambda) = c_1 + (N-1)\ln \bar{\alpha}_q + (N-r-1)\ln(\lambda - \lambda_a) + (N-2r)\ln\left(\frac{\lambda - \lambda_a^{-1}}{\lambda}\right) + \frac{N}{2\pi i}\oint \frac{d\lambda'}{\lambda - \lambda'} \ln g(t') - \frac{1}{2\pi i}\oint \frac{d\lambda'}{\lambda - \lambda'} U[t'] \tag{4.12}$$

where

$$g(t) = \prod_{j=1}^{r-1} (a - \omega^j t) \Big/ \prod_{j=r+1}^{N-1} (a - \omega^j t) \tag{4.13}$$

and

$$\begin{aligned} U[t] &= \sum_{j=1}^{N-1} \ln[(t_p - \omega^j t)^{jL} (t_{p'} - \omega^j t)^{jL} \tau_j(t)_{r=0}/\tau_{N-j}(\omega^j t)_{r=0}] \\ &= \sum_{j=1}^{N-1} \{jL \ln[(t_p - \omega^j t)(t_{p'} - \omega^j t)] \\ &\quad + (N-2j)\ln \tau_2(\omega^{j-1} t)_{r=0}\} \end{aligned} \tag{4.14}$$

The polynomial $\hat{S}(\lambda_q)$ is related to $\xi(\lambda_q)$ by (4.6), and so we get

$$\ln \hat{S}(\lambda_q) = c_3 + (N-1)\ln \bar{\alpha}_q + (N-r)\ln(\lambda_q - \lambda_a^{-1}) - r\ln(\lambda_q - \lambda_a) - (N-2r)\ln \lambda_q + \frac{N}{2\pi i}\oint \frac{d\lambda'}{\lambda_q - \lambda'} \ln g(t') - \frac{1}{2\pi i}\oint \frac{d\lambda'}{\lambda_q - \lambda'} U[t'] \tag{4.15}$$

(where c_3 is independent of q) for $|\lambda_q| > 1$. Finally, we can replace $1/(\lambda_q - \lambda')$ with $(\lambda_q + \lambda')/[2\lambda'(\lambda_q - \lambda')]$ while only adding on a q-independent constant, so if we let $\lambda' = e^{i\theta}$ and $t' = \Delta(\theta)$, then we get finally

$$\begin{aligned} \ln \hat{S}(\lambda_q) &= c_4 + (N-1)\ln \bar{\alpha}_q \\ &\quad - \tfrac{1}{2}[A(\lambda_q^{-1}, t_p) + A(\lambda_q^{-1}, t_{p'}) + B(\lambda_p, \lambda_q^{-1}) + B(\lambda_{p'}, \lambda_q^{-1})] \\ &\quad + (N-r)\ln(\lambda_q - \lambda_a^{-1}) - r\ln(\lambda_q - \lambda_a) \\ &\quad - (N-2r)\ln \lambda_q - NI(\lambda_q) \end{aligned} \tag{4.16}$$

where we let

$$I(\lambda_q) = \frac{1}{4\pi i}\oint \frac{d\lambda'}{\lambda'} \frac{\lambda_q + \lambda'}{\lambda_q - \lambda'} \ln g(t') \tag{4.17}$$

5. THE FREE ENERGY AND INTERFACIAL TENSION

Consider once again Eq. (2.45), taking $k=j$, $l=0$. From the limit $k' \to 0$, we see that when $|\lambda_q| > 1$ and L is large, the second term of the right-hand side exponentially dominates the first. Thus in the $L \to \infty$ limit, we have

$$T_q \Lambda_q^{(j,0)} \hat{T}_{\bar{q}j0} = H_{pq}^{(j)} \tau_{N-j}(\omega^j t_q) + \text{an exponentially smaller term} \tag{5.1}$$

We take the product of this equation over j running from 0 to $N-1$ and ignore the second term in (5.1), which is negligible for large L. From Eqs. (2.41) and (2.42) we have

$$\prod_{j=0}^{N-1} \Lambda_q^{(j,0)} = \left[\frac{(\omega\mu_p)^{N(N-1)/2}}{N^N (y_p - x_q)^N} \prod_{j=1}^{N-1} (y_{p'} - \omega^j y_q)^j (x_p - \omega^{N-j-1} y_q)^j \right]^L \tag{5.2}$$

and

$$\prod_{j=0}^{N-1} H_{pq}^{(j)} = \left[\frac{(\omega\mu_p)^{N(N-1)/2}}{(y_p^N - x_q^N)^N} \prod_{j=1}^{N-1} (t_p - \omega^{N-j-1} t_q)^j \right]^L \tag{5.3}$$

so using these and Eqs. (2.47) and (2.23) we derive an expression for the eigenvalues $T_q \hat{T}_q$:

$$\begin{aligned} (T_q \hat{T}_q)^N \hat{S}(\lambda_q)^2 &= e^{-\pi i L(N-1)(N-2)/3} (\lambda_q^2 / \lambda_p \lambda_{p'}^3)^{(N-1)L/2} \\ &\quad \times (\bar{\alpha}_q)^{2(N-1)} (g_{pq} g_{p'q})^{NL} D^N \\ &\quad \times \prod_{j=0}^{N-1} \tau_{N-j}(\omega^j t_q)^2 \prod_{j=1}^{N-1} [(t_q - \omega^j t_p)(t_q - \omega^j t_{p'})]^{-2(N-j)L} \end{aligned} \tag{5.4}$$

We take logarithms, substitute in the expression (4.16) for $\hat{S}(\lambda_q)$ from the previous section, and also note that

$$\begin{aligned} \sum_{j=0}^{N-1} \ln \tau_{N-j}(\omega^j t_q) &= -r \ln(a^N - t_q^N) + N \sum_{j=0}^{r-1} \ln(a - \omega^j t_q) \\ &\quad + \tfrac{1}{2}[C(\lambda_p, t_q) + C(\lambda_{p'}, t_q)] + L\pi i (N-1)(2N-1)/6 \end{aligned} \tag{5.5}$$

Equation (5.4) contains an unknown multiplicative term, the matrix D^N, which is independent of q. Because there is already an unknown term [the integration constant appearing in (4.16)], we can in fact write all terms which are independent of q as a single constant, $m_{pp'}$, which will be determined shortly. We write (5.4) as

$$N \ln(T_q \hat{T}_q) = m_{pp'} + E_{pq} + E_{p'q} + 2NF_q \tag{5.6}$$

The function E_{pq}, which is proportional to L, and therefore contributes to the bulk part of the partition function, was given in Eq. (2.58), and the function F_q, which is independent of L, is given by

$$F_q = -r \ln \mu_q - \ln[(\lambda_q - \lambda_a^{-1})/\lambda_q] + \sum_{j=0}^{r-1} \ln(a - \omega^j t_q) - I(\lambda_q) \tag{5.7}$$

This depends on p and p' only through a (and also λ_a), and contributes to the interfacial tension.

The constant $m_{pp'}$ is calculated as follows. Consider the $j = N-1$ case of Eq. (5.1); ignoring the exponentially small corrections (i.e., assuming L is large), the right-hand side is an exactly known function (the τ_{N-j} function in this case being unity). Applying the automorphism R of Eq. (2.7) to both sides, we find

$$T_{Rq} \hat{T}_q = (g_{pq}\, g_{qp})^L \qquad \text{and} \qquad T_q \hat{T}_{Rq} = (g_{p'q}\, g_{qp'})^L \tag{5.8}$$

From (5.6), we see that the constant $m_{pp'}$ is therefore given by

$$2m_{pp'} = l_p + l_{p'} - 2N(F_q + F_{Rq}) \tag{5.9}$$

where

$$l_p = NL \ln(g_{pq}\, g_{qp}) - E_{pq} - E_{p,Rq} \tag{5.10}$$

The function $E_{p,Rq}$ is defined by the analytic continuation formulas (2.54) and (2.55), and we find that $l_p + l_{p'} = 0$. The function F_{Rq} is defined in terms of the analytic continuation of $I(\lambda_q)$ to $|\lambda_q| < 1$, and we find that $I(\lambda_q)$ defined by (4.17) has the analytical continuation formula

$$I_{\text{ac}}(\lambda_q) + I(1/\lambda_q) = \ln\, g(t_q) \tag{5.11}$$

when $|\lambda_q| < 1$, and where $g(t_q)$ is given by (4.13). Hence we find that

$$m_{pp'} = -N(F_q + F_{Rq}) = -N \ln(-\lambda_a k'/k^2) \tag{5.12}$$

Let $2v_r = 2F_q + m_{pp'}$, so

$$v_r = r \ln \mu_q - \sum_{j=0}^{r-1} \ln(a - \omega^j t_q) + \ln[(\lambda_q - \lambda_a^{-1})/\lambda_q] + \tfrac{1}{2} \ln(-k'\lambda_a/k^2) + I(\lambda_q) \tag{5.13}$$

We can rewrite (5.13) so that it depends only on a, rather than on both a and λ_a. The contour of integration of $I(\lambda_q)$ as defined by (4.17) is around the unit circle in the λ' plane; if we change the variable of integration to t', where t' and λ' are related by (2.6), then we integrate around the branch cut between η and $1/\eta$ in the t' plane. Writing this as a line integral along one side of the cut, we can rewrite $I(\lambda_q)$ as

$$I(\lambda_q) = \frac{1}{\pi} \int_{\eta}^{1/\eta} dt'\, \Psi(\lambda_q, t') \frac{d}{dt'} \ln h\left(\frac{-\omega^{r/2} t'}{a}\right) + r \ln a - \ln\left(\frac{\lambda_q - \lambda_a^{-1}}{\lambda_q}\right) - \frac{1}{2} \ln\left(\frac{-\lambda_a k'}{k^2}\right) + \frac{1}{2} \ln h\left(\frac{-\omega^{r/2}}{a}\right) \tag{5.14}$$

where the functions $A(t)$, $h(t)$, and $\Psi(\lambda, t)$ are defined by (2.61)–(2.63), respectively. Letting $m = -\omega^{r/2}/a$, we find that v_r is given by (2.60). The variable a can take on any of the L values allowed by Eq. (3.22). Thus we have the following expression for the L largest eigenvalues of the transfer matrix:

$$N \ln(T_q \hat{T}_q) = L(E_{pq} + E_{p'q}) - 2v_r \tag{5.15}$$

To calculate the partition function, for large L, the sum in (2.27) will be dominated by these L bound-state eigenvalues. As $L \to \infty$, the sum becomes an integral over the allowed values of a, so we write

$$Z_r = \oint \rho(a)(T_q \hat{T}_q)^M \, da \tag{5.16}$$

where $\rho(a)$ is some distribution function which is independent of M and L. Noting that E_{pq} is independent of a, then (2.29) implies that the free energy is

$$-N\psi/k_B \mathscr{T} = \tfrac{1}{2}(E_{pq} + E_{p'q}) \tag{5.17}$$

[which reduces to (2.59), the analytical continuation of the ground-state eigenvalue of the system calculated in refs. 6 and 7, when $p = p'$]. Thus the interfacial tension is given by

$$e^{-Me_r/k_B \mathscr{T}} = \oint \rho(a)\, e^{-2Mv_r} \, da \tag{5.18}$$

For M large, this integral can be evaluated by the saddle-point method; the integral is dominated by the contribution from its saddle point, and so as $M \to \infty$ the integral is given by the value of its integrand at its saddle point, together with some multiplicative factors with which we are unconcerned.

We can now follow the arguments of refs. 1 and 2 regarding the location of the saddle point. In ref. 1 it was demonstrated that in the limit $k' \to 0$, the function v_r possesses a saddle point which is independent of p and p', and hence the integral (5.18) can be evaluated by deforming the contour of integration to pass through this saddle point. Assuming that this holds for general k', then we arrive at (2.64) for the interfacial tension. This depends on q but not on p or p'.

In ref. 2 the interfacial tension was considered in the scaling region, $k \to 0$, $k' \to 1$, $\eta \to 0$, where (2.64) simplifies to

$$\varepsilon_r / k_B \mathscr{T} = 4\rho \eta^{(N+2)/2} \sin(\pi r/N) \tag{5.19}$$

where

$$\rho = \frac{2}{\pi} \int_0^1 (1 - x^N)^{1/2}\, dx = 2\beta\left(\frac{1}{N}, \frac{1}{2}\right) \Big/ \pi(N+2) \tag{5.20}$$

is a beta function, and depends only on N (and not η). Hence in the scaling region the interfacial tension is independent of the horizontal rapidity q also. From (5.19) it follows that the critical exponent μ, defined by

$$\varepsilon_r \propto (\mathscr{T}_c - \mathscr{T})^{\mu} \qquad \text{as} \quad \mathscr{T} \to \mathscr{T}_c \tag{5.21}$$

is $\mu = 1/2 + 1/N$.

Au-Yang and Perk[8] consider the symmetric case $t_q = \omega^{1/2}$, $\lambda_q = [(1+k)/(1-k)]^{1/2}$, $t_p = 1$. Then Eq. (2.64) for the interfacial tension simplifies, becoming their Eq. (2.73). They also expressed the low-temperature interfacial tension as a dilogarithm integral, and wrote the next-order correction to the scaling region interfacial tension (5.19), thus finding the crossover exponent $\phi = 1/2 - 1/N$.

Finally, in both the low-temperature and scaling limits, the interfacial tensions satisfy the inequality $\varepsilon_j < \varepsilon_k + \varepsilon_l$, where $j = k + l \bmod N$. Hence in both of these limits the system is nonwetting, and is presumably so in the entire subcritical region $0 < k' < 1$. At zero teperature $k' = 0$ and criticality $k' = 1$, the system superwets, i.e., $\varepsilon_r = r\varepsilon_1$.[8]

6. SUMMARY

In summary, we have directly calculated the partition function of the general solvable chiral Potts model for a large rectangular lattice with the

skew-periodic boundary conditions (2.1) linking the left- and right-hand sides. This gives both the known free energy and the interfacial tension. The latter result agrees with that obtained previously by considering the superintegrable case and "Z-invariance" arguments.

To do this, we calculated a band of eigenvalues T_q of the transfer matrix. They can be thought of as excitations from the ground state of the case with periodic boundary conditions, and are characterized by a single string of r zeros of T_q. [At low temperatures it takes the simple form (3.8).] Presumably a general excitation (for large L) is simply an appropriate superposition of such elementary excitations. (Some work on excitations has been done by McCoy and Roan.[9,10])

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